Discontinuous Galerkin method for computing induced fields in superconducting materials

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***Abstract*— A discontinuous Galerkin method is proposed for computing the current density in superconductors characterized by the power law $J(E) \sim E^{\frac{1}{n}}$, with $n > 1$. This approach is applied to solve the non linear diffusion problem satisfied by E. An application example is given for a superconducting cylinder subjected to an external magnetic field. Results are compared to those given by the mixed FE-FV method and those obtained using a commercial software. Efficiency and robustness of the approach are illustrated on an example with $n = n(r)$.**

***Index Terms*— Superconductors, non linear diffusion, Discontinuous Galerkin method, interior penalty method, finite element (FE), finite volumes (FV).**

I. Introduction

The constitutive power law is widely used to characterize high temperature superconductors. It is written as:

$$\frac{J}{J_c} = \left(\frac{E}{E_c}\right)^{\frac{1}{n}} \tag{1}$$

where J is the current density, E the electric field, E_c the critical electric field, J_c the critical current density and n the power law exponent. The case $n = 1$ corresponds to a normal conductor, while $n = +\infty$ represents the critical state model suggested by Bean. Several numerical methods have been proposed to solve non linear diffusion problems resulting from Maxwell's equations [1], [2]. Their results are satisfying when n is uniform. Few of them are suited when n is large and models where n locally varies are uncommon.

In this paper we present a Discontinuous Galerkin (DG) method for computing induced fields in superconductors. We work on solving the non linear diffusion problem in terms of the electric field in order to determine the current density when n is large or locally varies. DG methods are well suited to treat discontinuous forms. They use high-order polynomials basis for reducing spurious oscillations. In addition they are naturally well adapted for parallel computing.

II. The differential system

In a two-dimensional setting where the magnetic induction depends only on two space variables ($B = (B_x, B_y)$), the electric field E and current density J have a single nonzero component, and can thus be treated as scalar fields. We set $u = E/E_c$, $\beta(u) = J/J_c$ and $c = \mu_0 J_C/E_c$. The superconductor sample has a vacuum magnetic permeability μ_0. Denoting the superconductor domain by Ω and its border by $\partial\Omega$, Maxwell's equations and the constitutive law (1) lead to the following non linear diffusion problem:

$$(S) \begin{cases} \dfrac{\partial \beta(u)}{\partial t} - \dfrac{1}{c}\triangle u = 0 \text{ in } \Omega \\ \overrightarrow{\nabla} u \cdot \overrightarrow{\nu} = C_b(t) \text{ on } \partial\Omega \end{cases} . \tag{2}$$

The system is established with a zero initial condition and the boundary condition on $\partial\Omega$ results from Faraday's law:

$$C_b(t) = E_c^{-1}\left(\frac{\partial B_y}{\partial t}, -\frac{\partial B_x}{\partial t}\right) \cdot \overrightarrow{\nu}, \tag{3}$$

where $\overrightarrow{\nu}$ is the outward normal vector.

III. Discontinuous Galerkin method

Let us consider a triangulation $\mathcal{I}_h = \bigcup K$ of the domain Ω. The Discontinuous Galerkin approach combines discretisation tools of finite element (FE) and finite volume (FV) methods. It consists in solving on each K the weak formulation of the system (2):

$$c\int_K \frac{\partial \beta(u)}{\partial t}\varphi dK - \int_K \nabla u \nabla \varphi dK - \int_\Gamma \nabla u \cdot \nu \varphi d\Gamma = 0 \tag{4}$$

where φ is a test function, $\Gamma = \partial K$ is an interface between two elements of $\mathcal{I}_h$ or a part of $\partial\Omega$.

A. FE discretization on each element

On each triangle $K \in \mathcal{I}_h$ an FE approximation space is defined. Its basis functions are polynomials of degree p. The number of nodes on K is given by $\frac{1}{2}(p+1)(p+2)$.

The discrete solution is written as $u^K = \sum_j u_j^K \varphi_j^K$, with u_j^K its value at node j. Since β is a liptschitz function, we assume that $\beta(uK) = \sum_j \beta(u_j^K)\varphi_j^K$. The mass matrix M^K is determined from the L^2 scalar product: $M_{ij}^K = \int_K \varphi_i^K \varphi_j^K dK$. The global mass matrix M^Ω is block diagonal.

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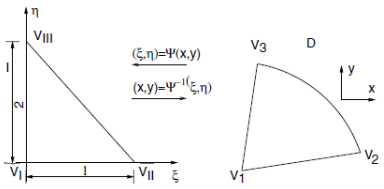

Fig. 1. Mapping of a simplex

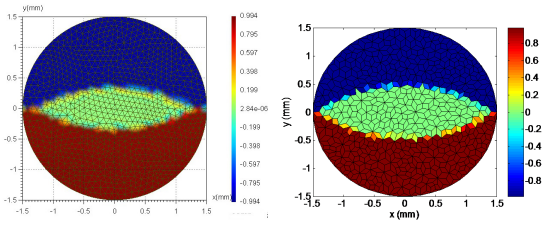

Fig. 2. Current density distribution at $t = 0.5s$ with n=200: (left) DG method with $p = 1$. (right): mixed FE-FV method

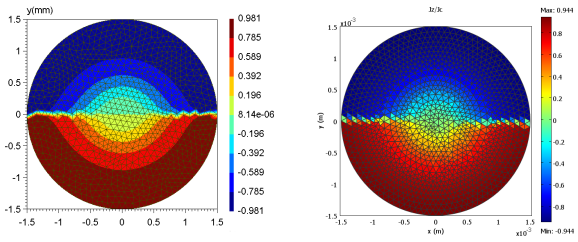

Fig. 3. Current density distribution at $t = 0.5s$, computed by DG method with $p = 4$ (left) and $H-$formulation (right)

B. Flux term on the interface of two elements

As in a FV method, the interface term $\int_\Gamma \nabla u \cdot \nu \varphi d\Gamma$ is treated by a numerical flux F which verifies $F_{K,L} = -F_{L,K}$, where K and L are neighboring elements. Its construction needs the following functions at the interface $K \cap L$: mean value $\{u\} = \frac{u_L + u_K}{2}$ and the jump $[|u|] = u_L - u_K$.

Many expressions of F have been proposed in the case of the Laplacian operator. We choose the expression of F based on the Non symmetric Interior Penality method (NIP). The NIP method consists in introducing a penalty term $\int_\Gamma \theta[|u|][|\varphi|]d\Gamma$, in order to guarantee continuity of u and ∇u at the interface [3]. The numerical flux is given by:

$$F \to \int_\Gamma \{\nabla u\} \cdot [|\varphi|]\nu^K d\Gamma + \int_\Gamma \theta[|u|] \cdot [|\varphi|]\nu^K d\Gamma \quad (5)$$

where θ is a positive parameter.

C. The discrete problem

Rules for evaluating the different terms of the weak formulation (4) exploit properties of mesh parametrization also called "mapping". The mapping is based on a bijective function Ψ such as $\Psi(x, y) = (\xi, \eta)$. This function allows to transform the physical space (x, y) to a parametric space (ξ, η) [4].

The basis functions of the parametric space are the linear combinations of $\xi^\alpha \eta^\beta$, where $\alpha + \beta \leq p$. In this space, derivative and integration operations are more convenient. The terms of the weak formulation are evaluated in the parametric space and mapped in the physical space.

After time discretization a discrete problem (6) is obtained on each K:

$$M_K \frac{\beta(u_{k+1}^K) - \beta(u_k^K)}{\delta t} = f_K(t, u_{k+1}^K) \quad (6)$$

where δ is the time step, u_k^K is solution at instant t_k, and $f^K(t, u_{k+1}^K)$ represents the discretization of the laplacian operator with numerical flux given in (5). A change of variable $v = \beta(u)$ is set and the current density $\beta(u)$ is calculated using a Newton iterative method.

IV. Numerical results

We consider a superconducting cylinder of radius $R = 1.5mm$, characterized by $J_c = 14.15A/mm^2$, $E_c = 10^{-4}V/m$. It is subjected to an external transverse magnetic field in the x direction, $B(t) = B_0 \sin(2\pi ft)$, with $f = 0.5Hz$, $T = 2s$ and $B_0 = 15mT$.

A. The case of constant n

For $n = 200$, the current density distribution is plotted at $t = T/4$. Notice that we did not get convergence using the $H-$formulation [1] implemented under the Comsol software. Fig. 2 presents a comparison of our result to that given by the mixed FE-FV method [2]. A good agreement is observed.

B. Example with non uniform n

It is well know that the n exponent is not a constant. It locally varies and becomes large in regions where temperature is close to $0K$.

In this example we suppose that $n(r) = n_0 \exp(4r/R)$, with $n_0 = 1$ and $r = \sqrt{x^2 + y^2}$. We note that $n(R) = 54$ and $n(0) = 1$.The comparison to the results issued from $H-$formulation show the validity of our approach. Fig.3 presents the current density distribution at $t = T/4$. Near the border n is large and J/J_c is close to 1. When approaching the center of the cylinder, n decreases and J/J_c becomes lower.

We will analyze these results further in the extended paper, and present a performance comparison between the new DG formulation, the FE-FV method and the standard FE technique.

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